Getting over it with April Grimoire

May 15, 2023

1 Statement

Suppose

$$\begin{array}{ccc} W & \xrightarrow{\beta'} & X \\ \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} \\ Y & \xrightarrow{\beta} & Z \end{array}$$

is a commutative diagram. Given a sheaf $\mathscr{F} \in \mathrm{Sh}(X)$, we can construct the push-pull map $\beta^{-1}\alpha_*\mathscr{F} \to (\alpha')^{-1}\beta'_*\mathscr{F}$ in the following two ways: (Denote by $\eta_\alpha, \epsilon_\alpha$ the unit and counit of the adjoint pair $\alpha^{-1} \dashv \alpha_*$)

1.

$$\begin{array}{ccccc} \mathrm{Hom}(\beta^{-1}\alpha_*\mathscr{F}, \alpha'_*(\beta')^{-1}\mathscr{F}) & \ni & \alpha'_*(\beta')^{-1}\epsilon_{\alpha,\mathscr{F}} \circ \eta_{\alpha',\beta^{-1}\alpha_*\mathscr{F}} & = & \mathrm{pushpull}_1 \\ \cong\uparrow & & \uparrow & & \\ \mathrm{Hom}((\alpha')^{-1}\beta^{-1}\alpha_*\mathscr{F}, (\beta')^{-1}\mathscr{F}) & & & & \\ \cong\uparrow & & & & \\ \mathrm{Hom}((\beta')^{-1}\alpha^{-1}\alpha_*\mathscr{F}, (\beta')^{-1}\mathscr{F}) & \ni & (\beta')^{-1}\epsilon_{\alpha,\mathscr{F}} & & \\ \uparrow & & \uparrow & & \\ \mathrm{Hom}(\alpha^{-1}\alpha_*\mathscr{F}, \mathscr{F}) & \ni & \epsilon_{\alpha,\mathscr{F}} & & \\ \cong\uparrow & & \uparrow & & \\ \mathrm{Hom}(\alpha_*\mathscr{F}, \alpha_*\mathscr{F}) & \ni & \mathrm{id}_{\alpha_*\mathscr{F}} & & \\ \uparrow & & \uparrow & & \\ \mathrm{Hom}(\mathscr{F}, \mathscr{F}) & \ni & \mathrm{id}_{\mathscr{F}} & & \end{array}$$

2.

$$\begin{array}{ccccc} \mathrm{Hom}(\beta^{-1}\alpha_*\mathscr{F}, \alpha'_*(\beta')^{-1}\mathscr{F}) & \ni & \epsilon_{\beta,\alpha'_*(\beta')^{-1}\mathscr{F}} \circ \beta^{-1}\alpha_*\eta_{\beta',\mathscr{F}} & = & \mathrm{pushpull}_2 \\ \cong\uparrow & & \uparrow & & \\ \mathrm{Hom}(\alpha_*\mathscr{F}, \beta_*\alpha'_*(\beta')^{-1}\mathscr{F}) & & & & \\ \cong\uparrow & & & & \\ \mathrm{Hom}(\alpha_*\mathscr{F}, \alpha_*\beta'_*(\beta')^{-1}\mathscr{F}) & \ni & \alpha_*\eta_{\beta',\mathscr{F}} & & \\ \uparrow & & \uparrow & & \\ \mathrm{Hom}(\mathscr{F}, \beta'_*(\beta')^{-1}\mathscr{F}) & \ni & \eta_{\beta',\mathscr{F}} & & \\ \cong\uparrow & & \uparrow & & \\ \mathrm{Hom}((\beta')^{-1}\mathscr{F}, (\beta')^{-1}\mathscr{F}) & \ni & \mathrm{id}_{(\beta')^{-1}\mathscr{F}} & & \\ \uparrow & & \uparrow & & \\ \mathrm{Hom}(\mathscr{F}, \mathscr{F}) & \ni & \mathrm{id}_{\mathscr{F}} & & \end{array}$$

2 Solution

Draw the two push-pull morphisms in the diagram

$$\begin{array}{ccc} \beta^{-1}\alpha_*\mathscr{F} & \xrightarrow{\eta_{\alpha',\beta^{-1}\alpha_*\mathscr{F}}} & \alpha'_*(\alpha')^{-1}\beta^{-1}\alpha_*\mathscr{F} = \alpha'_*(\beta')^{-1}\alpha^{-1}\alpha_*\mathscr{F} \\ \downarrow{\scriptstyle \beta^{-1}\alpha_*\eta_{\beta',\mathscr{F}}} & & \downarrow{\scriptstyle \alpha'_*(\beta')^{-1}\epsilon_{\alpha,\mathscr{F}}} \\ \beta^{-1}\alpha_*\beta'_*(\beta')^{-1}\mathscr{F} = \beta^{-1}\beta_*\alpha'_*(\beta')^{-1}\mathscr{F} & \xrightarrow{\epsilon_{\beta,\alpha'_*(\beta')^{-1}\mathscr{F}}} & \alpha'_*(\beta')^{-1}\mathscr{F} \end{array}$$

The top and bottom morphisms in the diagram above are morphisms in natural transformations, which encourages us to check the following diagrams, which make use of natuality:

$$\begin{array}{ccc}
\beta^{-1}\alpha_*\mathscr{F} & \xrightarrow{\eta_{\alpha',\beta^{-1}\alpha_*\mathscr{F}}} & \alpha'_*(\alpha')^{-1}\beta^{-1}\alpha_*\mathscr{F} \\
\Big\downarrow{\scriptstyle \beta^{-1}\alpha_*\eta_{\beta',\mathscr{F}}} & & \Big\downarrow{\scriptstyle \alpha'_*(\alpha')^{-1}\beta^{-1}\alpha_*\eta_{\beta',\mathscr{F}}} \\
\beta^{-1}\alpha_*\beta'_*(\beta')^{-1}\mathscr{F} & \xrightarrow{\eta_{\alpha',\beta^{-1}\alpha_*\beta'_*(\beta')^{-1}\mathscr{F}}} & \alpha'_*(\alpha')^{-1}\beta^{-1}\alpha_*\beta'_*(\beta')^{-1}\mathscr{F}
\end{array}$$

$$\begin{array}{ccc}
\beta^{-1}\beta_*\alpha'_*(\beta')^{-1}\alpha^{-1}\alpha_*\mathscr{F} & \xrightarrow{\epsilon_{\beta,\alpha'_*(\beta')^{-1}\alpha^{-1}\alpha_*\mathscr{F}}} & \alpha'_*(\beta')^{-1}\alpha^{-1}\alpha_*\mathscr{F} \\
\Big\downarrow{\scriptstyle \beta^{-1}\beta_*\alpha'_*(\beta')^{-1}\epsilon_{\alpha,\mathscr{F}}} & & \Big\downarrow{\scriptstyle \alpha'_*(\beta')^{-1}\epsilon_{\alpha,\mathscr{F}}} \\
\beta^{-1}\beta_*\alpha'_*(\beta')^{-1}\mathscr{F} & \xrightarrow{\epsilon_{\beta,\alpha'_*(\beta')^{-1}\mathscr{F}}} & \alpha'_*(\beta')^{-1}\mathscr{F}
\end{array}$$

Then there is connecting maps

$$\begin{array}{c}
\beta^{-1}\alpha_*\mathscr{F} \\
\Big\downarrow{\scriptstyle \beta^{-1}\eta_{\beta,\alpha_*\mathscr{F}}} \\
\beta^{-1}\beta_*\beta^{-1}\alpha_*\mathscr{F} \\
\Big\downarrow{\scriptstyle \beta^{-1}\beta_*\eta_{\alpha',\beta^{-1}\alpha_*\mathscr{F}}} \\
\beta^{-1}\beta_*\alpha'_*(\alpha')^{-1}\beta^{-1}\alpha_*\mathscr{F} \\
\Vert \\
\beta^{-1}\beta_*\alpha'_*(\beta')^{-1}\alpha^{-1}\alpha_*\mathscr{F}
\end{array}$$

$$\begin{array}{c}
\alpha'_*(\alpha')^{-1}\beta^{-1}\alpha_*\beta'_*(\beta')^{-1}\mathscr{F} \\
\Vert \\
\alpha'_*(\alpha')^{-1}\beta^{-1}\beta_*\alpha'_*(\beta')^{-1}\mathscr{F} \\
\Big\downarrow{\scriptstyle \alpha'_*(\alpha')^{-1}\epsilon_{\beta,\alpha'_*(\beta')^{-1}\mathscr{F}}} \\
\alpha'_*(\alpha')^{-1}\alpha'_*(\beta')^{-1}\mathscr{F} \\
\Big\downarrow{\scriptstyle \alpha'_*\epsilon_{\alpha',(\beta')^{-1}\mathscr{F}}} \\
\alpha'_*(\beta')^{-1}\mathscr{F}
\end{array}$$

$$\begin{array}{c}
\alpha'_*(\alpha')^{-1}\beta^{-1}\alpha_*\beta'_*(\beta')^{-1}\mathscr{F} \\
\Vert \\
\alpha'_*(\beta')^{-1}\alpha^{-1}\alpha_*\beta'_*(\beta')^{-1}\mathscr{F} \\
\Big\downarrow{\scriptstyle \alpha'_*(\beta')^{-1}\epsilon_{\alpha,\beta'_*(\beta')^{-1}\mathscr{F}}} \\
\alpha'_*(\beta')^{-1}\beta'_*(\beta')^{-1}\mathscr{F} \\
\Big\downarrow{\scriptstyle \alpha'_*\epsilon_{\beta',(\beta')^{-1}\mathscr{F}}} \\
\alpha'_*(\beta')^{-1}\mathscr{F}
\end{array}$$

Note that we provided two morphisms $\alpha'_*(\alpha')^{-1}\beta^{-1}\alpha_*\beta'_*(\beta')^{-1}\mathscr{F} \to \alpha'_*(\beta')^{-1}\mathscr{F}$. In fact, these two are identical: they both equal $\alpha'_*\epsilon_{\beta\alpha',(\beta')^{-1}\mathscr{F}}$. (There is similarly a counterpart for the first connecting map, but I'm too lazy to type it down.) It follows from the following general fact about adjoints:

Lemma. *For adjoint pairs*

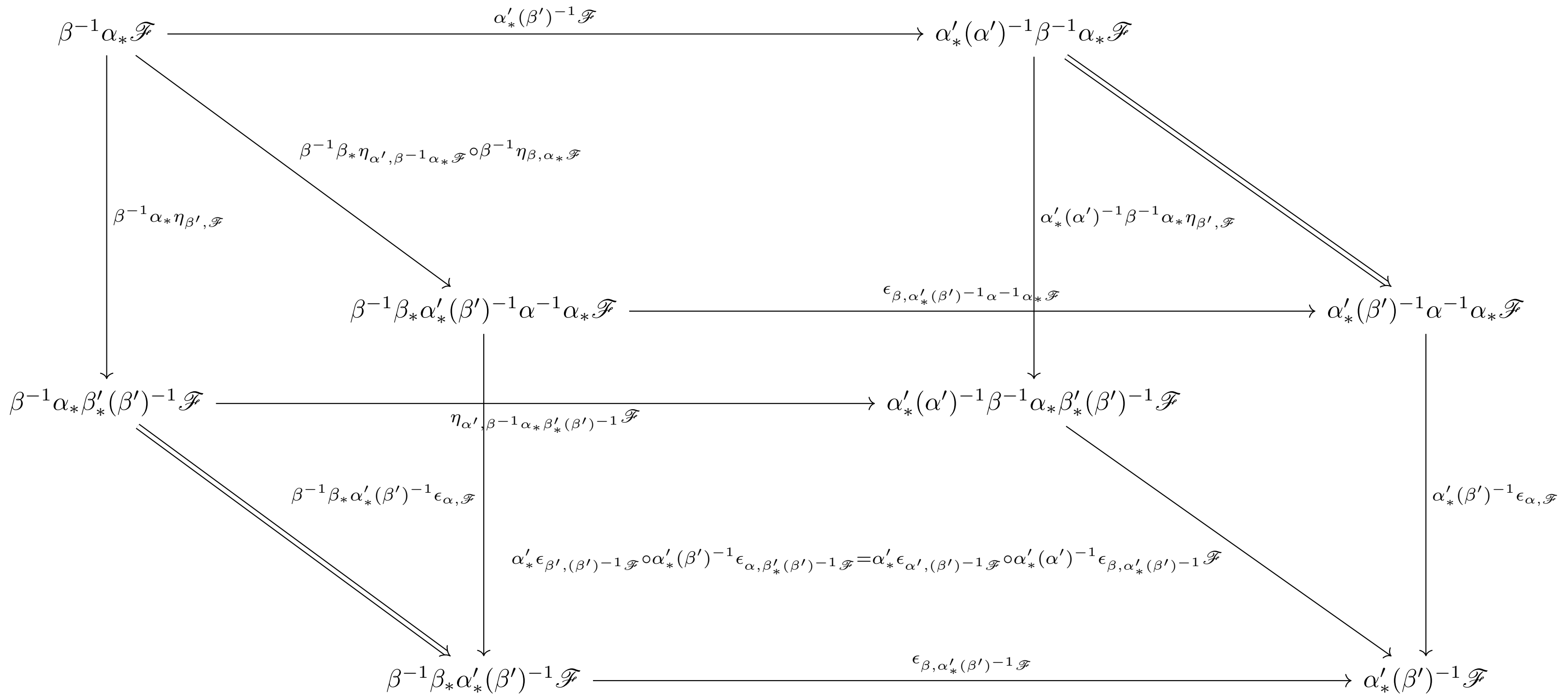
$$\mathscr{C} \underset{G_1}{\overset{F_1}{\rightleftarrows}} \mathscr{D} \underset{G_2}{\overset{F_2}{\rightleftarrows}} \mathscr{E}$$

Suppose $F_1 \dashv G_1$ *with unit and counit* η_1, ϵ_1, $F_2 \dashv G_2$ *with unit and counit* η_2, ϵ_2. *Then* $F_2F_1 \dashv G_1G_2$ *with unit and counit*

$$X \xrightarrow{\eta_{1,X}} G_1F_1X \xrightarrow{G_1\eta_{2,F_1X}} G_1G_2F_2F_1X$$

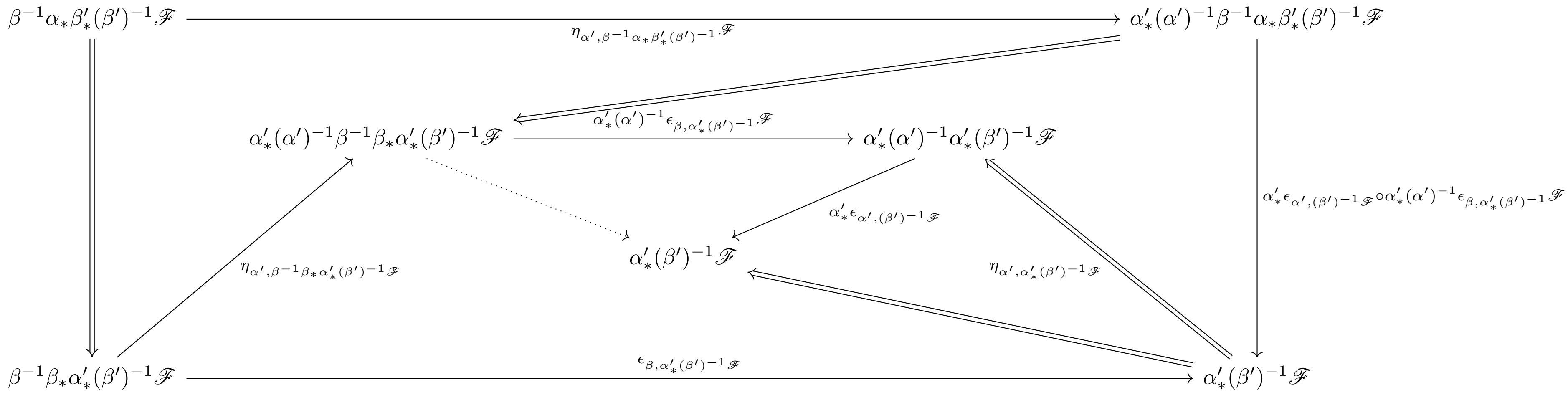
$$F_2F_1G_1G_2X \xrightarrow{F_2\epsilon_{1,G_2X}} F_2G_2X \xrightarrow{\epsilon_{2,X}} X$$

With these two maps we can connect the previous two natural transformation diagrams together into a huge diagram:



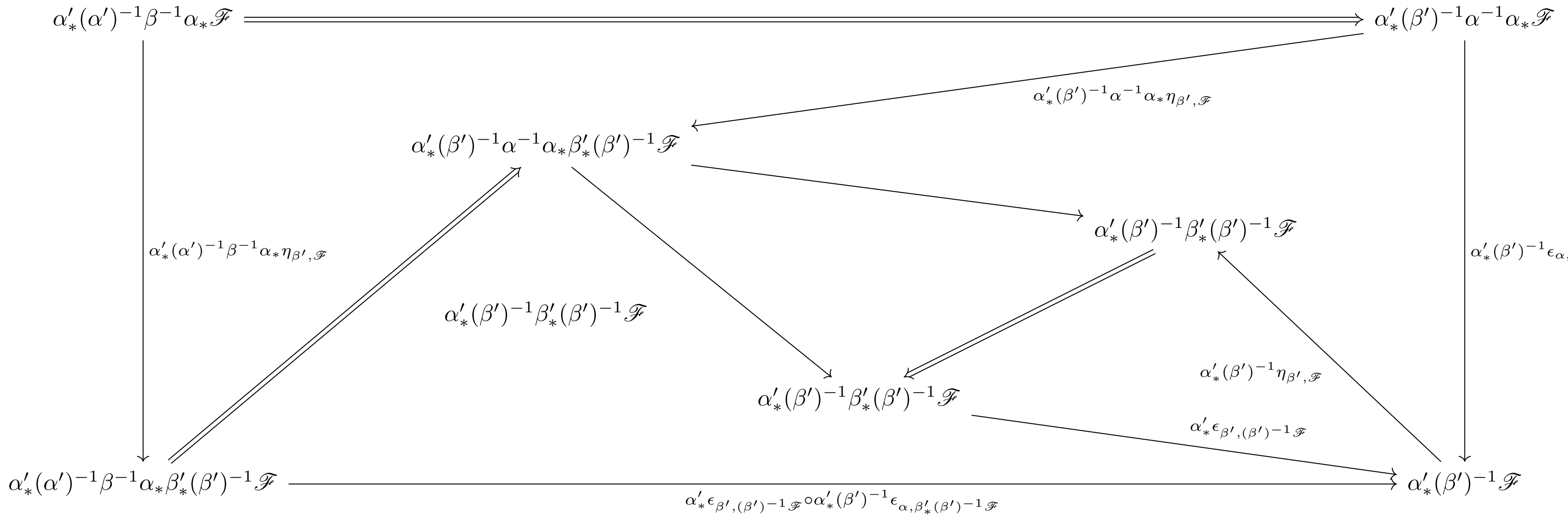
Viewing the diagram as a cube, the front and back faces are already proven to commute. If we can verify commutativity of the two more parrallelograms around $\alpha'_*(\beta')^{-1}\mathscr{F}$, the main result follows directly.

First we inspect the bottom parrallelogram:



Where the dotted arrow is just the composition to make the middle triangle commute. Hopefully the reader is able to make sense of why each cell commutes, somewhere using the equational property of unit and counit. I hope to make this clearer, however I am not good at typesetting commutative diagrams.

Likewise, we can verify the parallelogram in the right of the "cube" commutes:



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